

$$K(x, s; t) = \frac{\exp \left\{ -(4t)^{-1} [t^2 + \log^2 [(-x + (x^2 + 1)^{1/2})(s + (s^2 + 1))^{1/2}]] \right\}}{2(\pi t)^{1/2} (x^2 + 1)^{1/4} (s^2 + 1)^{1/4}} + R(x, s; t)$$

for  $x \leq s$ , interchanging  $x$  and  $s$  for  $s < x$ , where

$$\lim_{t \rightarrow +0} \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} R(x, s; t) f(s) ds \right| dx = 0$$

for every  $f \in L(-\infty, \infty)$ . The case  $b(x) = x^2 + 1$  gives a solution of a problem posed by A. Kolmogoroff in 1931.<sup>5</sup>

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<sup>2</sup> *J. Anal. Math.*, 3, 81-196, 1953/54.

<sup>3</sup> D. V. Widder, *The Laplace Transform* (Princeton: Princeton University Press, 1941), see p. 145.

<sup>4</sup> *Ibid.*, p. 161.

<sup>5</sup> *Math. Ann.*, 104, 415-58, 1931; see esp. p. 455.

## ON THE PURITY OF THE BRANCH LOCUS OF ALGEBRAIC FUNCTIONS

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1. Let  $V/k$  be an absolutely irreducible,  $r$ -dimensional normal algebraic variety and let  $K = k(V)$  be the function field of  $V/k$ ; here  $k$  denoted an arbitrary ground field. Let  $K^*$  be a finite separable algebraic extension of  $K$ , let  $k^*$  be the algebraic closure of  $k$  in  $K^*$ , and let  $V^*/k^*$  be a normalization of  $V$  in  $K^*$ . Let  $P^*$  be an arbitrary point of  $V^*$  (not necessarily algebraic over  $k$ ), and let  $P$  be the corresponding point of  $V$ . We denote by  $\mathfrak{o}$  the local ring of  $P$  on  $V/k$  and by  $\mathfrak{m}$  the maximal ideal of  $\mathfrak{o}$ . Let  $\mathfrak{o}^*$  and  $\mathfrak{m}^*$  have a similar meaning for  $P^*$  and  $V^*/k^*$ . It is well known that: (1)  $\mathfrak{o}^*\mathfrak{m}$  is a primary ideal, with  $\mathfrak{m}^*$  as associated prime ideal; (2) the residue field  $k^*(P^*) (= \mathfrak{o}^*/\mathfrak{m}^*)$  is a finite algebraic extension of the field  $k(P) (= \mathfrak{o}/\mathfrak{m})$ .

*Definition:* The point  $P^*$  is said to be unramified (with respect to  $V$ ) if the following conditions are satisfied:

- (a)  $\mathfrak{o}^*\mathfrak{m} = \mathfrak{m}^*$ ;
- (b)  $k^*(\mathfrak{p}^*)$  is a separable extension of  $k(P)$ .

In the contrary case  $P^*$  is said to be ramified (with respect to  $V$ ).

Note that, since  $K^*/K$  is separable,  $k^*/k$  is also separable, and hence (b) is equivalent to the condition:  $k(P^*)$  is separable over  $k(P)$ .

We fix an affine part  $V_a$  of  $V$  containing the point  $P$ . Let  $R = k[x_1, x_2, \dots, x_n]$  be the co-ordinate ring of  $V_a/k$  and let  $R^* = k^*[x_1, x_2, \dots, x_n, x_{n+1}, \dots, x_m]$  be

the integral closure of  $R$  in  $K^*$ . If  $V_a^*$  denotes the locus of the point  $(x_1, x_2, \dots, x_m)$  over  $k^*$ , then  $V_a^*$  is an affine representative of  $V^*$ , and  $P^* \in V_a^*$ . Let  $\{g_j(X_1, X_2, \dots, X_m); j = 1, 2, \dots, N\}$  be a basis of the ideal of  $V_a^*$  in  $k^*[X_1, \dots, X_m]$ .

PROPOSITION 1. *The point  $P^*$  is unramified if and only if the matrix*

$$\frac{\partial(g_1, g_2, \dots, g_N)}{\partial(X_{n+1}, X_{n+2}, \dots, X_m)} \quad (1)$$

has rank  $m - n$  at  $P^*$ .

*Proof:* We consider local  $k$ -derivations at  $P$  (on  $V$ ), with values in any given extension field  $\Omega$  of  $k(P)$ . By this we mean (see Zariski,<sup>2</sup> p. 43) mappings  $D$  of  $\mathfrak{o}$  into  $\Omega$  such that: (1)  $Dc = 0$  if  $c \in k$ ; (2)  $D(w_1 - w_2) = Dw_1 - Dw_2$ ; (3)  $D(w_1 w_2) = \bar{w}_1 Dw_2 + \bar{w}_2 Dw_1$ , where  $\bar{w}_i$  denotes the  $\mathfrak{m}$ -residue of  $w_i$ . Similarly, we shall consider local  $k^*$ -derivations at  $P^*$  (on  $V^*$ ).

Assume that  $P^*$  is unramified. Let  $D^*$  be a local  $k^*$ -derivation at  $P^*$ , on  $V^*$ , such that the induced local  $k$ -derivation  $D$  at  $P$  on  $V$  ( $D =$  restriction of  $D^*$  to  $\mathfrak{o}$ ) is zero. From  $\mathfrak{o}^* \mathfrak{m} = \mathfrak{m}^*$  it follows that  $D^* = 0$  on  $\mathfrak{m}^*$ . Now let  $w$  be any element of  $\mathfrak{o}^*$ , let  $\alpha$  be the  $\mathfrak{m}^*$ -residue of  $w$  ( $\alpha \in k(P^*)$ ), let  $\bar{F}(Z)$  be the minimal polynomial of  $\alpha$  over  $k(P)$  and let  $F(Z)$  be a polynomial in  $\mathfrak{o}[Z]$  such that the coefficients of  $\bar{F}(Z)$  are the  $\mathfrak{m}$ -residues of the coefficients of  $F(Z)$ . We have  $F(w) \in \mathfrak{m}^*$  and hence  $D^*(F(w)) = 0$ . On the other hand,  $D^*(F(w)) = \bar{F}'(\alpha)D^*w$ , and  $\bar{F}'(\alpha) \neq 0$ , since  $\bar{F}(Z)$  is a separable polynomial. Hence  $D^*w = 0$ . We have therefore shown that if  $D = 0$ , then also  $D^* = 0$ .

Now, given  $m$  quantities  $u_1, u_2, \dots, u_m$  in an extension field of  $k(P^*)$ , a necessary and sufficient condition that there should exist a local  $k$ -derivation  $D^*$  at  $P^*$ , on  $V^*$ , such that  $D^*x_i = u_i$  is that the following relations be satisfied (see Zariski,<sup>2</sup> p. 44):

$$\sum_{\nu=1}^m \left( \frac{\partial g_i}{\partial X_\nu} \right)_{P^*} u_\nu = 0, \quad i = 1, 2, \dots, N.$$

Our preceding result shows that if  $u_1 = u_2 = \dots = u_n = 0$ , then necessarily also  $u_{n+1} = u_{n+2} = \dots = u_m = 0$ . Therefore, the Jacobian matrix (1) must have rank  $m - n$  at  $P^*$ .

Conversely, assume that the matrix (1) has rank  $m - n$  at  $P^*$ . Let  $\xi_\nu$  be the  $\mathfrak{m}^*$ -residue of  $x_\nu$ , so that  $k(P) = k(\xi_1, \xi_2, \dots, \xi_n)$  and  $k^*(P^*) = k^*(\xi_1, \xi_2, \dots, \xi_m)$ . Our assumption on the matrix (1) implies (see Zariski,<sup>2</sup> Lemma 6, p. 27) that  $k^*(P^*)$  is a separable algebraic extension of  $k(P)$ . Thus condition (b) of the definition of unramified points is satisfied. Let, say,

$$\frac{\partial(g_1, g_2, \dots, g_{m-n})}{\partial(X_{n+1}, X_{n+2}, \dots, X_m)} \neq 0 \text{ at } P^*.$$

Then the  $m - n$  polynomials  $g_i(\xi_1, \xi_2, \dots, \xi_n, X_{n+1}, \dots, X_m)$  ( $i = 1, 2, \dots, m - n$ ) form a set of uniformizing parameters of the point  $(\xi_{n+1}, \xi_{n+2}, \dots, \xi_m)$  in the affine  $(m - n)$ -space over  $k^*(P)$ . It follows that if  $\{h_j(X_1, X_2, \dots, X_n); j = 1, 2, \dots\}$  is a set of uniformizing parameters of the point  $(\xi_1, \xi_2, \dots, \xi_n)$  in the affine  $n$ -space over  $k$  (and hence also in the affine space over  $k^*$ , since  $k^*/k$  is separable), then the polynomials  $g_i(X_1, X_2, \dots, X_m)$  ( $i = 1, 2, \dots, m - n$ ),  $h_j(X_1, X_2, \dots, X_n)$  ( $j =$

$1, 2, \dots$ ) form a set of uniformizing parameters of the point  $(\xi_1, \xi_2, \dots, \xi_m)$  in the affine  $m$ -space over  $k^*$ . This implies that the quantities  $h_j(x_1, x_2, \dots, x_n)$  form a basis of  $\mathfrak{m}^*$ , and, since they also form a basis of  $\mathfrak{m}$ , we have  $\mathfrak{m}^* = \mathfrak{o}^*\mathfrak{m}$ . Thus  $P^*$  is unramified.

The above proposition shows that the set of points of  $V^*$  which are ramified with respect to  $V$  is an algebraic variety (defined over  $k$ ). This variety is called the *branch locus of  $V^*/k^*$  (with respect to  $V/k$ )*. It will be denoted by  $\Delta$ .

2. The main object of this note is to prove the following result:

**PROPOSITION 2.** *If  $P^*$  is a point of  $\Delta$  such that the corresponding point  $P$  of  $V$  is a simple point of  $V/\mathfrak{k}$ , then  $\Delta$  is locally, at  $P^*$ , pure  $(r-1)$ -dimensional. (In particular, if  $V$  is a non-singular variety, then  $\Delta$  is a pure  $(r-1)$ -dimensional subvariety of  $V^*$ ).*

*Proof:* We shall give the proof only in the case in which  $k$  is either of characteristic zero or is a perfect field of characteristic  $p \neq 0$ . The generalization to non-perfect ground fields will be found in the note of N. Nagata which immediately follows the present note.

We first achieve a reduction to the case in which the ground field is algebraically closed. Let  $k'$  be the algebraic closure of  $k$  (in the universal domain). The varieties  $V$  and  $V^*$ , being absolutely irreducible, remain irreducible over  $k'$ . Since  $k$  (and therefore also  $k^*$ ) is perfect, the normal varieties  $V/k$  and  $V^*/k^*$  are absolutely normal, and thus  $V/k'$  is normal while  $V^*/k'$  is a normalization of  $V/k'$  in  $k'(V^*)$ . Again, since  $k^*$  is perfect, the prime ideal  $\mathfrak{J}(V_a/k^*)$  of any affine representative  $V_a^*$  of  $V^*/k^*$  remains prime under the ground field extension  $k^* \rightarrow k'$ , and thus a basis  $\{g_j(X_1, X_2, \dots, X_m); j = 1, 2, \dots, N\}$  of the ideal  $\mathfrak{J}(V_a^*/k^*)$  is also a basis of  $\mathfrak{J}(V_a^*/k')$ . It follows, therefore, from Proposition 1 that a point  $P^*$  of  $V^*$  is ramified with respect to  $V/k$  if and only if it is ramified with respect to  $V/k'$ . In other words: the branch locus  $\Delta$  of  $V^*$  with respect to  $V/k$  is independent of the choice of the (perfect) ground field  $k$ . On the other hand, any simple point  $P$  of  $V/k$  is absolutely simple and therefore remains simple under the ground field extension  $k \rightarrow k'$ . This shows that in the proof of our theorem we may replace  $k$  by  $k'$ .

Since the branch locus  $\Delta$  is algebraic, it will be sufficient to prove the proposition under the additional assumption that  $P$  is an algebraic point over  $k$ .

We therefore assume that  $k$  is an algebraically closed field and that  $P$  is rational over  $k$ .

We shall assume that the local  $P^*$ -component of  $\Delta$  is of dimension  $< r-1$ , and we shall show that in that case  $P^*$  is unramified.

We fix uniformizing parameters  $x_1, x_2, \dots, x_r$  of  $P$  on  $V/k$ . Let  $P'^*$  be any point of  $V^*$  such that  $\dim P'^*/k^* = r-1$  and such that  $P^*$  is a specialization of  $P'^*$  over  $k'$ . Let  $P'$  be the corresponding point of  $V$ . Then  $P$  is a specialization of  $P'$  over  $k$ , and—by our assumption— $P'^*$  is unramified with respect to  $V$ . The parameters  $x_i$  are also *uniformizing co-ordinates* of  $P'$  on  $V/k$ , i.e., they have the following two properties: (1)  $k[x_1, x_2, \dots, x_r]$  contains a uniformizing parameter of  $P'$  on  $V/k$ ; (2) the field  $k(P')$  is a separable algebraic extension of the field  $k(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n)$ , where the  $\bar{x}_i$  are the  $P'$ -residues of the  $x_i$ . Since  $P'^*$  is unramified, it follows that conditions (1) and (2) are satisfied also if  $P'$  and  $V$  are replaced by  $P'^*$  and  $V^*$ , respectively. Hence  $x_1, x_2, \dots, x_r$  are also uniformizing co-ordinates

of  $P^*$  on  $V^*/k$ . It follows that the derivations  $\frac{\partial}{\partial x_i}$  of  $K^*/k'$  (these derivations are defined since  $\{x_1, x_2, \dots, x_r\}$  is a separating transcendence basis of  $K^*/k'$ ) transform the local ring of  $P^*$  into itself. In particular, if  $w$  is any element of the local ring  $\mathfrak{o}^*$  of  $P^*$ , we have that  $\frac{\partial w}{\partial x_i}$  is regular at  $P^*$ . We have therefore shown that  $\frac{\partial w}{\partial x_i}$  has no polar prime divisor at the point  $P^*$ . Since  $P^*$  is a normal point, it follows that the  $r$ -functions  $\frac{\partial w}{\partial x_i}$  belong to  $\mathfrak{o}^*$ , for any  $w$  in  $\mathfrak{o}^*$ .

At this stage we shall separate the two cases  $p = 0$  and  $p \neq 0$ .

*Case  $p = 0$ .*—The fact that the partial derivations  $\partial/\partial x_i$  map  $\mathfrak{o}^*$  into itself permits us to define a homomorphic mapping

$$\phi: \mathfrak{o}^* \rightarrow k \langle x_1, x_2, \dots, x_r \rangle$$

of  $\mathfrak{o}^*$  into the power series ring  $k \langle x_1, x_2, \dots, x_r \rangle$ : for any  $w$  in  $\mathfrak{o}^*$  we define  $\phi(w)$  to be the power series

$$\sum_{(i)} c_{i_1 i_2 \dots i_r} x_1^{i_1} x_2^{i_2} \dots x_r^{i_r},$$

where  $c_{i_1 i_2 \dots i_r} = m^*$ -residue of

$$\frac{1}{i_1! i_2! \dots i_r!} \cdot \frac{\partial^{i_1 + i_2 + \dots + i_r} w}{\partial x_1^{i_1} \partial x_2^{i_2} \dots \partial x_r^{i_r}}.$$

The homomorphism  $\phi$  reduces to the identity on  $k[x_1, x_2, \dots, x_r]$ . Since the ideal  $\mathfrak{o}^*(x_1, x_2, \dots, x_r)$  contains a power of  $m^*$ , it follows at once that  $\phi$  can be extended (uniquely) to a homomorphism  $\bar{\phi}$  of the completion  $\bar{\mathfrak{o}}^*$  of  $\mathfrak{o}^*$  into  $k \langle x_1, x_2, \dots, x_r \rangle$ , and it is obvious that  $\bar{\phi}$  maps  $\bar{\mathfrak{o}}^*$  onto  $k \langle x_1, x_2, \dots, x_r \rangle$ . Now, since  $\{x_1, x_2, \dots, x_r\}$  is a system of parameters (in the sense of Chevalley) of  $\mathfrak{o}^*$ ,  $k \langle x_1, x_2, \dots, x_r \rangle$  can be identified with a subring of  $\bar{\mathfrak{o}}^*$  (Chevalley,<sup>1</sup> p. 702). It is obvious that  $\bar{\phi}$  reduces then to the identity on  $k \langle x_1, x_2, \dots, x_r \rangle$  (since  $\phi$  is the identity on  $k[x_1, x_2, \dots, x_r]$ ). It is also known that every element of  $\mathfrak{o}^*$  is integrally dependent on  $k \langle x_1, x_2, \dots, x_r \rangle$  (Chevalley,<sup>1</sup> p. 702) and that  $\bar{\mathfrak{o}}^*$  is an integral domain (Zariski<sup>3</sup>). Hence  $\bar{\phi}$  is an isomorphism. We have thus shown that  $\bar{\mathfrak{o}}^*$  is a regular ring, having  $x_1, x_2, \dots, x_r$  as regular parameters. Consequently, the same is true of  $\mathfrak{o}^*$ , showing that  $P^*$  is unramified.

*Case  $p \neq 0$ .*—The  $r$  partial derivations  $\frac{\partial}{\partial x_i}$  form a basis of the space of derivations of  $K^*/K^{*p}$ . Hence  $x_1, x_2, \dots, x_r$  form a  $p$ -basis of  $K^*$ . Now let  $w \in \mathfrak{o}^*$ . We can write  $w$  (uniquely) in the form

$$w = \sum A^{i_1 i_2 \dots i_r} x_1^{i_1} x_2^{i_2} \dots x_r^{i_r}, \quad A_{i_1 i_2 \dots i_r} \in K^*.$$

Using the fact that all the partial derivatives

$$\frac{\partial^{i_1 + i_2 + \dots + i_r} w}{\partial x_1^{i_1} \partial x_2^{i_2} \dots \partial x_r^{i_r}}$$

belong to  $\mathfrak{o}^*$ , we find at once that  $A^{i_1, i_2, \dots, i_r} \in \mathfrak{o}^*$ , and hence  $A_{i_1 i_2 \dots i_r} \in \mathfrak{o}^*$ , since

$\mathfrak{o}^*$  is integrally closed. It is clear that if  $w \in \mathfrak{m}^*$ , then  $A_{0,0}, \dots, c \in \mathfrak{m}^*$ , whence  $A^p_{0,0}, \dots, 0 \in \mathfrak{m}^{*p} \subset \mathfrak{m}^{*2}$ . It follows that

$$\mathfrak{m}^* \subset \sum_{i=1}^r \mathfrak{o}^* x_i + \mathfrak{m}^{*2},$$

and this implies that  $\mathfrak{m}^* = \sum_{i=1}^r \mathfrak{o}^* x_i = \mathfrak{o}^* \mathfrak{m}$ . This completes the proof.

3. Proposition 1 implies that  $P^*$  is unramified if and only if every local derivation  $D$  at  $P$  has at most one extension to a local derivation at  $P^*$ . We shall now prove

**PROPOSITION 3.** *If  $P^*$  is unramified (with respect to  $V$ ) then every local derivation  $D$  at  $P$  can be extended to a local derivation  $D^*$  at  $P^*$ . Hence  $P^*$  is unramified if and only if the vector space of local derivations at  $P^*$  is obtained from the vector space of local derivations at  $P$  by the extension  $k(P) \rightarrow k^*(P^*)$  of the field of scalars.*

*Proof:* We consider the completions  $\bar{\mathfrak{o}}$  and  $\bar{\mathfrak{o}}^*$  of  $\mathfrak{o}$  and  $\mathfrak{o}^*$ , respectively, and we set  $\bar{\mathfrak{m}} = \bar{\mathfrak{o}}\mathfrak{m}$ ,  $\bar{\mathfrak{m}}^* = \bar{\mathfrak{o}}^*\mathfrak{m}^*$ . Then  $\bar{\mathfrak{o}}$  and  $\bar{\mathfrak{o}}^*$  are local domains (since  $V$  and  $V^*$  are normal varieties; see Zariski<sup>3</sup>) and we have  $\bar{\mathfrak{m}}^* = \bar{\mathfrak{o}}^*\bar{\mathfrak{m}}$ . If then we set  $[k^*(P^*):k(P)] = g$ , then  $\bar{\mathfrak{o}}^*$  has an  $\bar{\mathfrak{o}}$ -basis consisting of  $g$  elements (Chevalley,<sup>1</sup> proof of Proposition 4, p. 695). As  $\bar{\mathfrak{o}}$ -basis of  $\bar{\mathfrak{o}}^*$  we can take any  $g$  element  $\bar{w}_i^*$  of  $\bar{\mathfrak{o}}^*$  whose  $\bar{\mathfrak{m}}^*$ -residues form a  $k(P)$ -basis of  $k^*(P^*)$ . Since  $k^*(P^*)$  is a separable algebraic extension of  $k(P)$ , it is a simple extension of  $k(P)$ . Let  $\alpha$  be a primitive element of  $k^*(P^*)/k(P)$  and let  $w^*$  be an element of  $\mathfrak{o}^*$  whose  $\mathfrak{m}^*$ -residue is  $\alpha$ . Then  $1, w^*, w^{*2}, \dots, w^{*(g-1)}$  form an  $\bar{\mathfrak{o}}$ -basis of  $\bar{\mathfrak{o}}^*$ , and  $w^*$  is a root of a monic polynomial  $f(X)$ , of degree  $g$ , with coefficients in  $\bar{\mathfrak{o}}$ . Since  $\bar{\mathfrak{o}}$  is also an integrally closed domain (by the theorem of analytical normality of normal varieties, Zariski<sup>4</sup>) and since every element of  $\bar{\mathfrak{o}}^*$  is integral over  $\bar{\mathfrak{o}}$ , the minimal (monic) polynomial of  $w^*$  over the quotient field of  $\bar{\mathfrak{o}}$  has coefficients in  $\bar{\mathfrak{o}}$ . This minimal polynomial cannot therefore be of degree  $< g$ , since otherwise  $\alpha$  would be a root of a polynomial of degree  $< g$ , with coefficients in  $k(P)$ . Hence  $f(X)$  is the minimal polynomial of  $w^*$ , and thus  $1, w^*, w^{*2}, \dots, w^{*(g-1)}$  are linearly independent over  $\bar{\mathfrak{o}}$ .

Now let  $D$  be any local derivation at  $P$ , on  $V$ . This derivation can be extended to a local derivation  $\bar{D}$  of the completion  $\bar{\mathfrak{o}}$  of  $\mathfrak{o}$  by setting, for any element  $\bar{y}$  of  $\bar{\mathfrak{o}}$ :  $\bar{D}\bar{y} = D\bar{y}$ , where  $y$  is any element of  $\mathfrak{o}$  such that  $\bar{y} - y \in \bar{\mathfrak{m}}^2$ . Since  $\bar{\mathfrak{m}}^2 \cap \mathfrak{o} = \mathfrak{m}^2$ , it follows at once that  $\bar{D}\bar{y}$  depends only on  $\bar{y}$ , and one immediately verifies that  $\bar{D}$  is a local derivation in  $\bar{\mathfrak{o}}$ . Now, if  $\bar{y}^*$  is any element of  $\bar{\mathfrak{o}}^*$ , we write  $\bar{y}^* = \bar{y}_0 + \bar{y}_1 w^* + \dots + \bar{y}_{g-1} w^{*(g-1)}$ , where the  $\bar{y}_i$  are uniquely determined elements of  $\bar{\mathfrak{o}}$ , and we set

$$\bar{D}^* \bar{y}^* = \sum_{i=0}^{g-1} \bar{D}\bar{y}_i \cdot \alpha^i + \left( \sum_{i=0}^{g-1} i \bar{\eta}_i \alpha^{i-1} \right) \bar{D}^* w^*,$$

where  $\bar{\eta}_i$  is the  $\bar{\mathfrak{m}}$ -residue of  $\bar{y}_i$  and where  $\bar{D}^* w^*$  is the element of  $k^*(P^*)$  determined by the relation

$$f'(\alpha) + \bar{D}^* w^* = 0.$$

Here, if

$$f(X) = \sum_{i=0}^g \bar{b}_i X^i, \quad (\bar{b}_i \in \bar{\mathfrak{o}}),$$

we set

$$f^{\bar{D}}(X) = \sum_{i=0}^g (\overline{Db_i}) X^i,$$

$$f(X) = \sum_{i=0}^g \beta_i X^i, \quad \beta_i = \overline{m}\text{-residue of } b_i.$$

Since  $f(X)$  is a separable polynomial,  $f'(\alpha)$  is different from zero and hence  $\bar{D}^*w^*$  is well defined. It is a straightforward matter to show that the mapping  $\bar{D}^*$  of  $\bar{o}^*$  onto  $k^*(P^*)$  is a local derivation of  $\bar{o}^*$  and that  $\bar{D}^*$  is an extension of  $\bar{D}$ . If we now set  $D^* = \text{restriction of } \bar{D}^* \text{ to } \bar{o}^*$ , we see that  $D^*$  is a local derivation at  $P^*$ , on  $V^*$ , and that  $D^*$  is an extension of  $D$ .

The second part of the proposition is an immediate consequence of the first part and of Proposition 1.

<sup>1</sup> C. Chevalley, "On The Theory of Local Rings," *Ann. Math.*, **44**, 690-708, 1943.

<sup>2</sup> O. Zariski, "Analytical Irreducibility of Normal Varieties," *Ann. Math.*, **49**, 352-61, 1948.

<sup>3</sup> O. Zariski, "The Concept of a Simple Point of an Abstract Algebraic Variety," *Trans. Am. Math. Soc.*, **62**, 1-52, 1942.

<sup>4</sup> O. Zariski, "Sur la normalité analytique des variétés normales," *Ann. Inst. Fourier*, **2**, 161-64, 1950.

## REMARKS ON A PAPER OF ZARISKI ON THE PURITY OF BRANCH-LOCI\*

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In the proof of Zariski of the purity of branch loci,<sup>1</sup> the following fact, which was proved by him, is one of the key points of his proof:

Let  $P$  be a simple spot over a field  $k$  and let  $x_1, \dots, x_n$  be a regular system of parameters of  $P$ . Assume that  $P/\sum x_i P = k$ . If a normal spot  $Q$  dominates  $P$  and is a ring of quotients of a finite separable integral extension of  $P$  and if every partial derivation  $\partial/\partial x_i$  can be extended to an integral derivation<sup>2</sup> of  $Q$ , then  $Q$  is unramified over  $P$ .

If the assumption that  $Q$  is normal is omitted, then the above becomes false. In the present note, we shall show at first that if  $k$  is of characteristic zero, then the assumption that  $Q$  is normal is unnecessary. In fact, we shall prove the following theorem which is a generalization of it:

**THEOREM.** *Let  $P$  be a spot over a field  $k$  of characteristic zero and let  $r$  be the dimension of the function field  $L$  of  $P$ . Assume that there exist algebraically independent elements  $x_1, \dots, x_r$  of  $P$  over  $k$  such that the partial derivations  $\partial/\partial x_i$  ( $i = 1, \dots, r$ ) can be extended to integral derivations of  $P$ . Then  $P$  is a regular local ring with a uniformizing co-ordinates  $x_1, \dots, x_r$ , namely, if we denote by  $\mathfrak{p}$  the intersection of the maximal ideal  $\mathfrak{m}$  of  $P$  with  $k[x_1, \dots, x_r]$ , then  $P$  is unramified over  $k[x_1, \dots, x_r]_{\mathfrak{p}}$ .*

On the other hand, Zariski proved the purity of branch loci only for the case of